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REPLY
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This article is a reply to a comment by Kong et al. (2019), https://doi.org/10.1029/2018WR024665.

Key Points:
- For the original Dupuit-Boussinesq theory, zero volumetric discharge at the upstream yields zero groundwater table or zero seepage velocity
- Groundwater table can be of physical significance only upon a nonnegative value
- A constant rainfall recharge determines a linear discharge distribution

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Reply to comment by Kong et al. on "Appropriate Boundary Condition for Dupuit-Boussinesq Theory on the Steady Groundwater Flow in an Unconfined Sloping Aquifer With Uniform Recharge"

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Abstract The article aims to respond a comment made on our paper about appropriate boundary condition for the original Dupuit-Boussinesq theory for two-dimensional steady groundwater flow in an unconfined sloping aquifer with uniform rainfall recharge. To respond to the comments arguing the existence of lateral groundwater flows and negative groundwater table, clarifications are made for our analysis focusing on two-dimensional groundwater flow without considering lateral effects by using the original and classical approximate theory.

The authors would like to thank Kong and other colleagues for their interest in and a comment article (Kong et al., 2019) made on our research paper (Wu et al., 2018) investigating appropriate boundary conditions for the original Dupuit-Boussinesq theory for steady two-dimensional groundwater flow in an unconfined sloping aquifer with uniform rainfall recharge. We are delighted to read the comments (Kong et al., 2019) pointing out shortcomings of the classical Dupuit-Boussinesq theory for general sloping aquifers and providing possible directions for broadening the scope of analytical analysis on shallow groundwater flow in unconfined aquifers. We do agree with Kong and other colleagues that an unsaturated zone is of importance for general and natural aquifers that are not the investigation focus of our paper (Wu et al., 2018). Instead, our paper's main purpose is to demonstrate one important analytical issue of determination of an appropriate boundary condition, which has not drawn much attention and comprehensively discussed in the past. Through this reply, we would like to further clarify our work to address the comments (Kong et al., 2019) arguing the effects of lateral flow and negative groundwater table.

The original Dupuit-Boussinesq theory (hereafter denoted as the original theory), or called *hydraulic groundwater theory*, is widely used for modeling groundwater flow in an unconfined aquifer under the assumptions that the capillary effect is insignificant and the aquifer is shallow (Brutsaert, 2005). These assumptions successfully simplify analysis on relating problems but yield some constraints that pressure is hydrostatic and seepage flow is independent of the vertical coordinate of the slope coordinate system. These constraints do not always match flow conditions in real hillslope aquifers that are insufficiently shallow, unneglectable for capillary effect, and possess inhomogeneous porosity. However, the original theory still can provide approximate solutions very close to ones obtained by a more complete formulation, so it is the method of choice in many investigations (Brutsaert, 2005) or for further developments of improvement (e.g., Hilberts & Troch, 2005, Kong et al., 2016, Luo et al., 2018, Troch et al., 2003). But, as is clearly pointed out in Wu et al. (2018), it is still lacking of an efficient way to determine appropriate boundary conditions when applying this original and classical theory. Based on its simplicity and relevance to hillslope hydrology analysis, the original theory is still worthy of our investigation focus as a foundation for further analytical development.

Here we would like to clarify again our analysis to possibly address the comment arguing that *the seepage velocity must be set to zero to satisfy the zero discharge boundary condition at the upstream boundary*. Without proposing any new formulation as well as source and/or sink in the aquifer, we were focusing on revealing the characteristics of the dynamical system of the original theory on a two-dimensional steady flow. Within the scope of physical significance defined in problems of our specific interest, we have interpreted dynamical behaviors of the value and the gradient of groundwater table and demonstrated parametric conditions on the phase planes when considering aquifers under three different groundwater hillslope flow numbers $\beta = L \tan \alpha / H$=10, 1.0, and 0.1, where α is the inclination and H and L are characteristic aquifer's thickness and length, respectively. For precise discussion, all aquifers considered in our analysis were categorized by βs not only by the inclination. As is selected for comprehensively demonstrating the dynamical system, the phase plane method successfully reveals the original theory possesses the behavior that either zero groundwater table or zero seepage velocity do separately exist in aquifers of certain settings and under certain rainfall recharges. To emphasize again, in our analysis, another important advantage of the phase plane analysis is that this method parametrically analyzes the original theory to obtain analytical solutions of groundwater table and its gradient everywhere from the downstream boundary without any presumption of upstream boundary beforehand. The analysis finally testifies the existence of two possible boundary conditions at the upstream when using the original theory. Also, for solutions of physical significance, the original theory always retains a nonnegative and real-valued groundwater table under a criterion,

$$0<I\leq\frac{k_0 \tan\alpha}{4}, \tag{1}$$

where I is the rainfall recharge and k_0 is the aquifer's hydraulic conductivity. Explanations refer to (16) and following paragraph in Wu et al. (2018).

For the other comments regarding the discharge distribution, we would like to respond as follows. To fit the steady state assumption, the original theory possesses a linear distribution of discharge relating to rainfall recharge from the downstream outlet to the upstream with a nonnegative groundwater table. This condition is also self-evident in our analysis of the classical and original Dupuit-Boussinesq theory for two-dimensional groundwater flow in unconfined sloping aquifers with constant recharge. To remind again, as no source and/or sink is considered, the lateral flow may not be able to be correctly modeled using the original theory.

The responses are summarized to possibly address the comments (Kong et al., 2019). Again, we thank the comments (Kong et al., 2019) for allowing us to take this chance of reply to the comments (Kong et al., 2019) to further clarify the applicability and validness of our analysis.

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